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The flow field near the centre of a rolled-up vortex sheet

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Most of the existing methods for calculating the inviscid flow past a delta wing with leading-edge vortices are based on slender-body theory. When these vortices are represented by rolled-up vortex sheets in an otherwise irrotational flow, some of the assumptions of slender-body theory are violated near the centres of the spirals. The aim of the present report is to describe for the vortex core an alternative method in which only the assumption of a conical velocity field is made. An asymptotic solution valid near the centre of a rolled-up vortex sheet is derived for incompressible flow. Further asymptotic solutions are determined for two-dimensional flow fields with vortex sheets which vary with time in such a manner that the sheets remain similar in shape. A particular two-dimensional flow corresponds to the slender theory approximation for conical sheets.

1. Introduction

Observation shows that, when a slender delta wing is placed at incidence in a stream, the fluid separates from the surface along lines near the leading edges. Shear layers, springing from these lines, form in the fluid and roll up (in opposite senses on the two halves of the wing) into spiral vortices which lie above the wing and inboard of the leading edges. Although viscous effects play some part in this flow, mainly in determining the position of the separation lines and in the inner part of the core, various attempts (Legendre 1952; Mangler & Smith 1959; Smith 1966*a*, *b*) have been made to calculate this flow on the basis of an inviscid flow model, in particular for a wing with sharp leading edges where the position of the separation lines is known.

The theory developed by Legendre (1952), Mangler & Smith (1959), Smith (1966*a*) is based on slender-body theory. However, in the vortex core velocities have been measured (see e.g. Earnshaw 1961) which differ considerably from those of the main stream which implies that it is doubtful whether slender theory is applicable near the centre of the core. We therefore aim in this paper to examine the flow near the centre of a coiled vortex sheet by a method which avoids the assumption of small perturbations.

Experiments suggest that a delta wing of aspect ratio less than two produces a vortex sheet which even in subsonic flow is approximately conical over the forward part of the wing, away from the trailing edge and outside the immediate neighbourhood of the apex. Therefore, it was decided to consider the flow as conical, i.e. the velocity components are assumed to be constant along rays from the apex of the delta wing.

In the present paper we consider only a narrow region around the vortex core and derive an asymptotic expansion of the velocity field near the centre, based on conical incompressible flow. We avoid thus the difficulties† which would arise if one were to try to calculate the entire flow field past a delta wing by conical incompressible theory.

We further derive an asymptotic expansion for the velocity field near the centre of a conical vortex sheet when the assumptions of slender theory are made. The two results will be compared to show the consequences of the slenderness assumption. The equations derived by slender theory for conical sheets are the same as for the two-dimensional flow field of a vortex sheet which grows linearly with time. This is a special case of the two-dimensional flow fields of endless rolled-up vortex sheets which vary with time such that their shapes remain similar. A certain family of two-dimensional similar vortex sheets which grow with time has already been investigated by Prandtl (1922). Another family, including the one with linear growth, is considered in this paper.

We ignore viscous effects and assume that the vorticity is concentrated on a thin sheet so that the flow between the turns of the sheet is irrotational. The task is then to find solutions of the equation for the velocity potential which satisfy the boundary conditions at a free vortex sheet.

We attack the problem by introducing two stream functions and transforming to co-ordinates X, Y such that the two faces of the vortex sheet become lines $Y = \text{const.}$, say $Y = 0$ and $Y = \gamma$. The problem is then to solve three non-linear first-order partial differential equations subject to the boundary conditions at $Y = 0$ and $Y = \gamma$. We derive only the leading terms of formal asymptotic expansions in terms of the distance from the centre of the sheet. The convergence of these expansions is not investigated, nor are the solutions they represent necessarily unique. The conical and the time-dependent problems are treated separately.

2. Conical vortex sheets

2.1. *General equations*

We investigate the flow field near the core of a rolled-up conical vortex sheet. For steady flows the sheet must be a stream surface, which means that the velocity vector is tangential to the sheet. We assume constant total head for the core region. Then the condition that the sheet cannot sustain a pressure difference between its two faces requires that the magnitude of the velocity is the same on opposite faces of the sheet, but the directions of the velocity vectors may differ.

The velocity field must satisfy the continuity equation. We ignore compressi-

† Germain (1955) has shown that the assumption of a wholly conical incompressible flow must lead to the existence of singularities in the flow field outside the wing and the vortex sheets originating at the wing; as a consequence the solution is not uniquely defined.

bility effects, then the continuity equation can be written in spherical polar co-ordinates (R, ψ, θ (see figure 1)) as

$$\frac{\partial}{\partial R}(R^2 v_R \sin\psi) + \frac{\partial}{\partial \psi}(R v_\psi \sin\psi) + \frac{\partial}{\partial \theta}(R v_\theta) = 0. \tag{1}$$

This equation is satisfied automatically by the velocity components which are derived from any pair of functions $\Psi_1(R, \psi, \theta)$ and $\Psi_2(R, \psi, \theta)$ by means of the following equations:

$$R^2 v_R \sin\psi = \frac{\partial \Psi_1}{\partial \psi}\frac{\partial \Psi_2}{\partial \theta} - \frac{\partial \Psi_1}{\partial \theta}\frac{\partial \Psi_2}{\partial \psi}, \tag{2}$$

$$R v_\psi \sin\psi = \frac{\partial \Psi_1}{\partial \theta}\frac{\partial \Psi_2}{\partial R} - \frac{\partial \Psi_1}{\partial R}\frac{\partial \Psi_2}{\partial \theta}, \tag{3}$$

$$R v_\theta = \frac{\partial \Psi_1}{\partial R}\frac{\partial \Psi_2}{\partial \psi} - \frac{\partial \Psi_1}{\partial \psi}\frac{\partial \Psi_2}{\partial R}. \tag{4}$$

It follows from (2) to (4) that

$$\left(v_R \frac{\partial}{\partial R} + \frac{V_\psi}{R}\frac{\partial}{\partial \psi} + \frac{v_\theta}{R\sin\psi}\frac{\partial}{\partial \theta}\right)\Psi_i = 0$$

so that Ψ_1 and Ψ_2 are constant along streamlines.

There is some freedom in the choice of stream functions Ψ_1 and Ψ_2 for a given three-dimensional flow. We make use of this to choose them for the conical problem so that

$$\Psi_1 = R^2 f(\psi, \theta), \tag{5}$$

$$\Psi_2 = g(\psi, \theta). \tag{6}$$

$g(\psi, \theta) = \text{const.}$ represents the intersection of the conical stream surfaces with the sphere $R = 1$. The vortex sheet is thus represented by a curve $g(\psi, \theta) = \text{const.}$

From (2) to (6) we obtain the following relations between the velocity components and the functions f and g:

$$v_R \sin\psi = f_\psi g_\theta - f_\theta g_\psi, \tag{7}$$

$$\left.\begin{aligned} v_\psi \sin\psi &= -2f g_\theta, \\ v_\theta &= 2f g_\psi. \end{aligned}\right\} \tag{8}$$

The subscripts to v represent the velocity components; subscripts to all other letters denote partial derivatives.

Since we assume that we are dealing with a potential flow, a function $\Phi(R, \psi, \theta)$ exists such that

$$v_R = \frac{\partial \Phi}{\partial R},$$

$$v_\psi = \frac{1}{R}\frac{\partial \Phi}{\partial \psi},$$

$$v_\theta = \frac{1}{R\sin\psi}\frac{\partial \Phi}{\partial \theta}.$$

For conical flow, the potential function is of the form

$$\Phi = R\phi(\psi, \theta) \tag{9}$$

and

$$v_R = \phi, \tag{10}$$

$$v_\psi = \phi_\psi, \tag{11}$$

$$v_\theta = \frac{1}{\sin\psi}\phi_\theta. \tag{12}$$

We introduce the variable

$$\xi = -\log\tan\tfrac{1}{2}\psi, \tag{13}$$

i.e.

$$d\xi = -\frac{d\psi}{\sin\psi},$$

$$\sin\psi\cosh\xi = 1.$$

With ξ and θ as the independent variables we obtain from (7) to (13) three partial differential equations for ϕ, f and g:

$$\phi = \cosh^2\xi(f_\theta g_\xi - f_\xi g_\theta), \tag{14}$$

$$\phi_\xi = 2fg_\theta, \tag{15}$$

$$\phi_\theta = -2fg_\xi. \tag{16}$$

In order to obtain relatively simple equations for the boundary conditions at the vortex sheet, we introduce new independent variables X, Y so that

$$\theta = \theta(X, Y),$$

$$\xi = X.$$

Then

$$\frac{\partial}{\partial\theta} = \frac{1}{\theta_Y}\frac{\partial}{\partial Y},$$

$$\frac{\partial}{\partial\xi} = \frac{\partial}{\partial X} - \frac{\theta_X}{\theta_Y}\frac{\partial}{\partial Y}.$$

Now we choose

$$Y = g,$$

so that

$$g_\theta = \frac{1}{\theta_Y},$$

$$g_\xi = -\frac{\theta_X}{\theta_Y}.$$

The three equations (14) to (16) take the form

$$\theta_Y\phi = -\cosh^2 X f_X, \tag{17}$$

$$\theta_Y\phi_X = 2f[1+(\theta_X)^2], \tag{18}$$

$$\phi_Y = 2f\theta_X. \tag{19}$$

We note that this sytem of differential equations is equivalent to equations (14) to (16) if and only if θ_Y is neither zero nor infinite. This condition is equivalent to the statement that no conical stream surface is either tangential or normal to a circular cone ψ = const. In this way the range of possible solutions is restricted.

By the transformation of the ψ, θ variables to the X, Y variables, the region between the turns of the spiral vortex sheet (projected on the unit sphere $R = 1$) is mapped on a strip $0 < Y < \gamma$ in the (X, Y)-plane so that the lines $Y =$ const. correspond to the projections of the streamlines on the unit sphere, with $Y = 0$ and $Y = \gamma$ corresponding to the two faces of the sheet (see figure 1). The lines

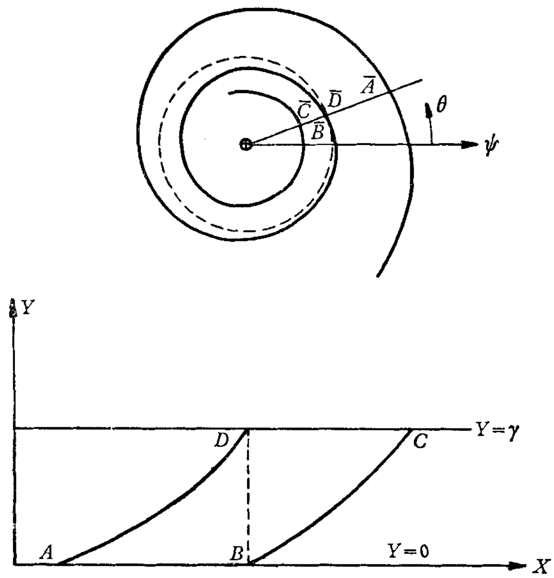

FIGURE 1. Notation.

$X =$ const. correspond to the projections of the cones $\psi =$ const. We consider the special case where $\psi = 0$, i.e. $\xi \to \infty$, represents the centre of the vortex sheet; then the vortex core, $\psi < \psi_0$ say, is represented by the semi-infinite strip $X > -\log\tan\frac{1}{2}\psi_0$, $0 \leqslant Y \leqslant \gamma$. The region $ABCD$ in the (X, Y)-plane is mapped on the region $\bar{A}\bar{B}\bar{C}\bar{D}$ in the (ψ, θ)-plane, with the straight line BD representing a circle $\psi =$ const., and the curves AD and BC representing lines $\theta =$ const. For two points $\bar{B}$ and $\bar{D}$ on either side of the vortex sheet, $\xi = X$ has the same value but the angle θ differs by 2π. The geometric boundary condition at the sheet reads therefore

$$\Delta\theta(X) \equiv \theta(X, \gamma) - \theta(X, 0) = -2\pi. \tag{20}$$

When this condition is satisfied, then we obtain a one-to-one mapping between the (ψ, θ) and the (X, Y)-planes.

For the velocity components we obtain the equations:

$$v_R = \phi, \tag{21}$$

$$v_\psi = -\cosh\xi\,\phi_\xi = -\cosh X \frac{1}{1+(\theta_X)^2}\phi_X, \tag{22}$$

$$v_\theta = \cosh\xi\,\phi_\theta = \cosh X \frac{1}{1+(\theta_X)^2}\phi_X\theta_X. \tag{23}$$

The magnitude V of the velocity is given by

$$V^2 = v_R^2 + v_\psi^2 + v_\theta^2$$
$$= \phi^2 + \cosh^2 X \frac{1}{1+(\theta_X)^2} (\phi_X)^2.$$

The pressure condition, $\Delta V^2(X) = 0$, leads thus to the condition

$$\Delta\left[\phi^2 + \cosh^2 X \frac{1}{1+(\theta_X)^2} (\phi_X)^2\right] = 0.$$

Since it follows from equation (20) that $\Delta\theta_X = 0$, we may replace this by

$$\phi_m \Delta\phi + \frac{\cosh^2 X}{1+(\theta_X)^2} \frac{d\phi_m}{dX} \frac{d\Delta\phi}{dX} = 0 \tag{24}$$

with $$2\phi_m(X) = \phi(X,\gamma) + \phi(X,0).$$

2.2. *Derivation of an asymptotic expansion for the velocity field*

We intend to obtain an asymptotic solution of the equations (17) to (19) which satisfies the conditions (20) and (24) for the inner part of the vortex core, i.e. for small ψ or large X. The presence of the term $\cosh X$ in equations (17) and (24) suggests that we try to obtain an asymptotic expansion of a solution in powers of e^X. We therefore seek a solution in which the functions $\theta(X, Y)$, $\phi(X, Y)$ and $f(X, Y)$ for large X are expressed as series in e^X:

$$\theta = \theta_0(X, Y)\, e^X + \theta_1(X, Y) + \theta_2(X, Y)\, e^{-X} + \ldots, \tag{25}$$

$$\phi = \phi_0(X, Y) + \phi_1(X, Y)\, e^{-X} + \ldots, \tag{26}$$

$$f = f_0(X, Y)\, e^{-2X} + f_1(X, Y)\, e^{-3X} + \ldots. \tag{27}$$

We assume that the coefficients and their first derivatives are bounded for large X or tend to infinity like X^m. This excludes such functions as $\sin e^X$.

It follows from equations (17) and (18) that

$$\frac{\phi_X}{\phi}\frac{f_X}{f} = -\frac{1+\theta_X^2}{\cosh^2 X}.$$

This equation together with the assumptions about the coefficients in the series (26) and (27) require that the series for θ begins with $\theta_0(X, Y)\, e^X$. The boundary condition (20) requires that $\theta_0(X, 0) = \theta_0(X, \gamma)$. This and the assumption that θ_Y must not vanish for finite X and $0 \leqslant Y \leqslant \gamma$ are satisfied if θ_0 is independent of Y.

We satisfy the boundary condition if

$$\theta_1(X, Y) = -2\pi \frac{Y}{\gamma} + h_1(X)\dagger \tag{28}$$

and $$\theta_n(X, 0) = \theta_n(X, \gamma) \quad \text{for} \quad n \neq 1. \tag{29}$$

From equations (18) and (19) it follows that ϕ_0 is a function of X only and that ϕ_{1Y} does not vanish identically if ϕ_{0X} does not vanish identically. It follows from

† There exists a multitude of functions $\theta_1(X, Y)$ which satisfy the conditions. The particular choice in equation (28) has no influence on the velocity potential when this is expressed in terms of ψ and θ.

equation (17) that $f(X, Y)$ behaves like e^{-2X} if $\phi(X, Y)$ behaves like an algebraic function.

To determine the functions $\theta_0(X)$, $h_1(X)$, $\theta_2(X, Y)$, ... $\phi_0(X)$, $\phi_1(X, Y)$, ... we insert the expansions (25) to (27) into the differential equations (17) to (19) and the boundary condition (24) and consider in each equation the leading terms. Then the following set of equations is obtained:

$$f_{0X} - 2f_0 = \frac{8\pi}{\gamma}\phi_0,$$

$$2f_0(\theta_0 + \theta_{0X})^2 = -\frac{2\pi}{\gamma}\phi_{0X},$$

$$2f_0(\theta_0 + \theta_{0X}) = \phi_{1Y},$$

$$4(\theta_0 + \theta_{0X})^2\phi_0\Delta\phi_1 = \phi_{0X}(\Delta\phi_1 - \Delta\phi_{1X}).$$

It can be shown that these equations are satisfied by

$$f_0 = -\frac{4\pi}{\gamma}C^2(X + \delta + \tfrac{1}{2}),$$

$$\phi_0 = C^2(X + \delta),$$

$$\phi_1 = -\frac{4\pi}{\gamma}C^2 Y(X + \delta + \tfrac{1}{2})^{\frac{1}{2}} + g_1(X),$$

$$\theta_0 = e^{-X}\left[\int_{X_0}^{X}\frac{e^{\xi}}{2(\xi + \delta + \frac{1}{2})^{\frac{1}{2}}}d\xi + h_0\right],$$

where C is a non-zero constant, δ is an arbitrary constant, and h_0 is a constant related to the origin of the θ-scale. It can be shown that, with the assumptions made about the coefficients in equations (25) to (27), the solution is unique.

To determine the function $g_1(X)$ above and $h_1(X)$ in (28) we consider in the differential equations (17) to (19) and in the boundary condition (24) the terms which are one order smaller than the leading terms. This procedure leads to the following set of equations:

$$F(X) = (X + \delta + \tfrac{1}{2})^{\frac{1}{2}}, \tag{30}$$

$$f_{1X} - 3f_1 = -2C^2(2F^2 - 1)\theta_{2Y} + \frac{8\pi}{\gamma}\left(-\frac{4\pi}{\gamma}C^2FY + g_1\right),$$

$$f_1 = 2C^2F^2\theta_{2Y} - \frac{4\pi}{\gamma}F^2\left(\frac{dg_1}{dX} - g_1 - 4C^2F\frac{dh_1}{dX} + \frac{2\pi}{\gamma}YC^2\frac{2F^2 - 1}{F}\right),$$

$$\frac{f_1}{F} - \frac{8\pi}{\gamma}C^2F^2\frac{dh_1}{dX} = \phi_{2Y},$$

$$\Delta\theta_2 = 0,$$

$$\Delta\phi_{2X} - \frac{2F^2 + 1}{2F^2}\Delta\phi_2 + 2\pi\left[\frac{2F^2 - 1}{F}\frac{dg_1}{dX} - \frac{2F^2 + 1}{F}g_1 - 4C^2(2F^2 - 1)\frac{dh_1}{dX} + \pi C^2\left(4F^2 + \frac{1}{F^2}\right)\right] = 0.$$

It can be shown that this set of equations is uniquely satisfied by

$$g_1 = 2\pi C^2 F,$$
$$h_1 = \pi,$$
$$f_1 = \frac{8\pi}{\gamma} C^2 F(2F^2+1)\left(\frac{2\pi}{\gamma} Y - \pi\right),$$
$$\phi_2 = C^2\left[2(2F^2+1)\left(\frac{2\pi}{\gamma} Y - \pi\right)^2 + g_2(X)\right],$$
$$\theta_2 = \frac{6F^2+1}{2F}\left(\frac{2\pi}{\gamma} Y - \pi\right)^2 + h_2(X).$$

To determine the functions $g_2(X)$ and $h_2(X)$, we have again to go back to the differential equations and the boundary conditions and solve the set of equations derived by considering the terms of the next lower order. After some lengthy calculations we obtain the following solution for $g_2(X)$ and $h_2(X)$:

$$g_2 = -2F^2 - 1 + \tfrac{4}{3}\pi^2 - 4\pi^2 F^2 e^{2F^2} \operatorname{ei}(2F^2), \tag{31}$$

where
$$\operatorname{ei}(\lambda) = \int_\lambda^\infty \frac{e^{-t}}{t}\, dt,$$

$$h_2 = -\frac{1}{4F} - \frac{\sqrt{\pi}}{4} e^{F^2} \operatorname{erfc}(F) + \frac{\pi^2}{3}\left[\frac{7}{2F} + 2F - \frac{3\sqrt{\pi}}{2} e^{F^2} \operatorname{erfc}(F)\right]$$
$$+ 2\pi^2 e^{F^2} \int_F^\infty dz \int_z^\infty \frac{e^{3z^2-4t^2} - e^{z^2-2t^2}}{tz^2}\, dt, \tag{32}$$

where
$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_\lambda^\infty e^{-t^2} dt.$$

We have not derived further terms in the expansions since the algebra involved is very complicated.

2.3. *Discussion of the expansion*

Summarizing the results, we quote the equations for the shape of the streamlines and for the potential function:

$$\theta(X, Y) = \int_{X_0}^X \frac{e^t}{2F(t)} dt - \frac{2\pi}{\gamma} Y + \pi - h_0 + \left[\frac{6F^2+1}{2F}\left(\frac{2\pi}{\gamma} Y - \pi\right)^2 + h_2(X)\right] e^{-X} + \dots, \tag{33}$$

$$\frac{\phi(X, Y)}{C^2} = X + \delta - 2F\left(\frac{2\pi}{\gamma} Y - \pi\right) e^{-X} + \left[2(2F^2+1)\left(\frac{2\pi}{\gamma} Y - \pi\right)^2 + g_2(X)\right] e^{-2X} + \dots, \tag{34}$$

where $F(X)$ is defined in (30), $h_2(X)$ and $g_2(X)$ are given in (31), (32). X is equal to ξ which is related to the cone angle ψ by equation (13) and a conical stream surface is a line $Y = $ const. and the two faces of the sheet are given by $Y = 0$ and $Y = \gamma$.

The potential difference across the sheet is

$$\frac{|\Delta\Phi|}{RC^2} = 4\pi F\, e^{-X} - 2\pi\left\{\frac{4F^2+1}{2F} + \frac{\pi^2}{3}\left(20F^3 - 16F - \frac{10}{F}\right) + \frac{2\pi^2}{F}\,[(4F^2+1)\,e^{2F^2}\,\mathrm{ei}\,(2F^2) - e^{4F^2}\,\mathrm{ei}\,(4F^2)]\right\} e^{-3X} + \dots$$
$$= 4\pi \tan\tfrac{1}{2}\psi(-\log\tan\tfrac{1}{2}\psi + \delta + \tfrac{1}{2})^{\frac{1}{2}} + O(\psi^3)$$
$$= 2\pi\psi(-\log\tfrac{1}{2}\psi + \delta + \tfrac{1}{2})^{\frac{1}{2}}[1 + O(\psi^2)]. \tag{35}$$

The e^{-3X} term has been obtained from the set of equations from which the functions $h_2(X)$ and $g_2(X)$ were derived.

By eliminating Y from equations (33) and (34) we can express ϕ as a function of θ and of X, i.e. of ψ. To do this, we introduce the notation $\theta_s(\psi)$ for the value of θ which the inside face of the vortex sheet has for a given ψ. $\theta_s(\psi)$ results from equation (33) for $Y = 0$. $\theta_s(\psi)$ is a single-valued function in accordance with our assumption that conical stream surfaces $g =$ const. are never tangential to the cones $\psi =$ const.

The relation between the velocity field and the potential function is given by equations (21) to (23). We obtain:

$$\frac{v_R}{C^2} = -\log\tan\tfrac{1}{2}\psi + \delta - 2\tan\tfrac{1}{2}\psi(-\log\tan\tfrac{1}{2}\psi + \delta + \tfrac{1}{2})^{\frac{1}{2}}[\theta_s(\psi) - \theta - \pi] + O(\psi^2\log\psi)$$
$$= -\log\tfrac{1}{2}\psi + \delta - \psi(-\log\tfrac{1}{2}\psi + \delta + \tfrac{1}{2})^{\frac{1}{2}}[\theta_s(\psi) - \theta - \pi] + O(\psi^2\log\psi), \tag{36}$$

$$\frac{v_\psi}{C^2} = -2F^2 e^{-X} - 2F(2F^2-1)\left(\frac{2\pi}{\gamma}Y - \pi\right)e^{-2X} + \dots$$
$$= -\psi(-\log\tfrac{1}{2}\psi + \delta + \tfrac{1}{2}) - \psi^2(-\log\tfrac{1}{2}\psi + \delta)(-\log\tfrac{1}{2}\psi + \delta + \tfrac{1}{2})^{\frac{1}{2}}[\theta_s(\psi) - \theta - \pi] + O(\psi^3(\log\psi)^2), \tag{37}$$

$$\frac{v_\theta}{C^2} = F + (2F^2-1)\left(\frac{2\pi}{\gamma}Y - \pi\right)e^{-X} + \left\{-\frac{1}{4F} + \pi^2\left(\tfrac{4}{3}F^3 + 2F + \frac{7}{6F}\right) - \left(2F^3 + 2F - \frac{1}{2F}\right)\left(\frac{2\pi}{\gamma}Y - \pi\right)^2 - (\pi^2/F)[(4F^2+1)\,e^{2F^2}\,\mathrm{ei}\,(2F^2) - e^{4F^2}\,\mathrm{ei}\,(4F^2)]\right\}e^{-2X} + \dots$$
$$= (-\log\tfrac{1}{2}\psi + \delta + \tfrac{1}{2})^{\frac{1}{2}} + \psi(-\log\tfrac{1}{2}\psi + \delta)[\theta_s(\psi) - \theta - \pi] + O(\psi^2(\log\psi)^{\frac{3}{2}}). \tag{38}$$

In these formulae θ varies between $\theta_s(\psi) - 2\pi$ and $\theta_s(\psi)$.

To simplify the comparison with other work, we express the velocity field in terms of cylindrical polar co-ordinates x, r, θ.

$$v_x = v_R\cos\psi - v_\psi\sin\psi,$$

$$\frac{v_x}{C^2} = -\log\frac{r}{2x} + \delta - \frac{r}{x}\left(-\log\frac{r}{2x} + \delta + \tfrac{1}{2}\right)^{\frac{1}{2}}\left[\theta_s\left(\frac{r}{x}\right) - \theta - \pi\right] + \dots, \tag{39}$$

$$v_r = v_R\sin\psi + v_\psi\cos\psi,$$

$$\frac{v_r}{C^2} = -\frac{1}{2}\frac{r}{x} - \left(\frac{r}{x}\right)^2\left[-\log\frac{r}{2x} + \delta + 1\right]\left(-\log\frac{r}{2x} + \delta + \tfrac{1}{2}\right)^{\frac{1}{2}}\left[\theta_s\left(\frac{r}{x}\right) - \theta - \pi\right] + \dots, \tag{40}$$

$$\frac{v_\theta}{C^2} = \left(-\log\frac{r}{2x} + \delta + \tfrac{1}{2}\right)^{\frac{1}{2}} + \frac{r}{x}\left[-\log\frac{r}{2x} + \delta\right]\left[\theta_s\left(\frac{r}{x}\right) - \theta - \pi\right] + \dots. \tag{41}$$

The shape of the section of the vortex sheet by a plane x = const. is given by

$$\frac{d\theta_s}{d\left(\dfrac{r}{x}\right)} = \frac{-1}{\left(\dfrac{r}{x}\right)^2\left(-\log\dfrac{r}{2x}+\delta+\frac{1}{2}\right)^{\frac{1}{2}}}+O\left(\log\frac{r}{x}\right). \tag{42}$$

We note that the axial and the circumferential velocity components tend to infinity when we approach the centre of the spiral whilst the radial velocity component tends to zero. The first term in each of the equations for the velocity components is independent of θ. The velocity field arising from the first term of the asymptotic expansion is thus continuous across the sheet and axially symmetric.

It is interesting to compare the velocity field with that obtained for an axially symmetrical conical flow with continuously distributed vorticity. Hall (1961) and Ludwieg (1962) have investigated this flow for slender cones, i.e. for small values of r/x. Their solution reads

$$v_x = C_1\{-\log(r/x)+C_2\},$$
$$v_r = -\tfrac{1}{2}C_1(r/x),$$
$$v_\theta = C_1(-\log(r/x)+\tfrac{1}{2}+C_2)^{\frac{1}{2}},$$

where C_1 and C_2 are arbitrary constants. A comparison of this solution with equations (39) to (41) gives the noteworthy result that for our particular case the leading terms of the velocity components for a potential flow with the vorticity concentrated along a vortex sheet are the same as for an axisymmetric flow with distributed vorticity.

We note that all the terms that have been found in the present asymptotic solution are determined by the choice of the two constants C and δ, just as the axially symmetric solution is determined by the two constants C_1 and C_2.

The present approach of term by term evaluation of the asymptotic expansion cannot provide information about its convergence. The expansion is certainly not valid when a stream surface g = const. (e.g. the vortex sheet) is tangential to a cone ψ = const. and therefore the derivative $g_\theta = 1/\theta_Y$ vanishes. The transformation to the X, $Y = g$ co-ordinates breaks down in this region. Experimental evidence (see e.g. Earnshaw 1961) and numerical solutions in Smith (1966*a*) suggest that the vortex sheet from the leading edge of a delta wing is tangential to such a circular cone at certain points well away from the centre of the sheet. Whether infinitely many further points of tangency arise as the centre is approached cannot be determined by either means.

3. Time-dependent two-dimensional vortex sheets

3.1. *General equations*

We investigate the flow field near the centre of a vortex sheet in two-dimensional incompressible flow which varies with time. The sheet is a discontinuity of the velocity field which is carried along by the flow. For a free vortex sheet in inviscid flow the vorticity of a fluid element does not change with time, therefore, the sheet consists of the same particles at all times. This statement about the way in which the sheet moves with the fluid leads to the geometric boundary condition. The second boundary condition is again derived from the requirement that no pressure force must act on the sheet. For a time-dependent flow, Bernoulli's equation for the static pressure contains the derivative of the potential function with respect to time. Therefore, the condition that the pressures on the two faces of the sheet be the same does not require that the magnitude of the velocities be the same.

The velocity field must again satisfy the continuity equation. Let r and θ be polar co-ordinates and t denote the time co-ordinate, then the continuity equation reads

$$\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_\theta}{\partial \theta}=0. \tag{43}$$

Since we consider only irrotational flow, a potential function $\Phi(r,\theta,t)$ exists, such that

$$v_r=\frac{\partial \Phi}{\partial r}, \tag{44}$$

$$v_\theta=\frac{1}{r}\frac{\partial \Phi}{\partial \theta}. \tag{45}$$

Equation (43) becomes the Laplace equation

$$\Phi_{rr}+\frac{1}{r}\Phi_r+\frac{1}{r^2}\Phi_{\theta\theta}=0.$$

It is possible to solve the problem by means of the theory of complex functions since Φ is the real part of the complex function $\Phi+i\Psi$ where Ψ is the stream function.

We intend to solve the problem by a method which is similar to the one used in §2. Since the problem is concerned with particle paths, we introduce two functions $\Psi_1(r,\theta,t)$ and $\Psi_2(r,\theta,t)$ which are constant along particle paths. If Ψ_1 and Ψ_2 satisfy the equations

$$r=\Psi_{1r}\Psi_{2\theta}-\Psi_{1\theta}\Psi_{2r}, \tag{46}$$

$$rv_r=\Psi_{1\theta}\Psi_{2t}-\Psi_{1t}\Psi_{2\theta}, \tag{47}$$

$$v_\theta=\Psi_{1t}\Psi_{2r}-\Psi_{1r}\Psi_{2\theta}, \tag{48}$$

then equation (43) is satisfied identically. Furthermore,

$$r\left(\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial \theta}\right)\Psi_i=0,$$

i.e. Ψ_1 and Ψ_2 are constant along a particle path. The values of Ψ_1 and Ψ_2 can be taken as 'labels' which identify a particle. If we give to the particles forming the vortex sheet at one time $t=t_0$ one common label $\Psi_2=$ const. then the sheet is at all times defined by $\Psi_2=$ const. The inside face of the vortex sheet is denoted by $\Psi_2=0$ and the outside by $\Psi_2=\gamma$.

We consider here only self similar flow fields for which the streamline pattern at any time t is similar to the pattern at time $t=t_0$. This means that the velocity components can be written as functions of $r/h(t)$ and θ with a scale factor which depends only on t.

We introduce the variables $\tau = t$ and $r^* = r/h(t)$ and choose Ψ_2 to be independent of τ. Then it follows from equations (44) to (48) that Ψ_1, Ψ_2 and Φ are expressible in the form†

$$\Psi_1 = h^2(\tau)\, f(r^*, \theta), \tag{49}$$

$$\Psi_2 = g(r^*, \theta), \tag{50}$$

$$\Phi = h(dh/dt)\,[\phi(r^*, \theta) + \tfrac{1}{2}r^{*2}]. \tag{51}$$

From equations (46), (49), (50) we obtain

$$r^* = f_{r^*} g_\theta - f_\theta g_{r^*}, \tag{52}$$

from equations (44), (47), (49) to (52)

$$r^* \phi_{r^*} = -2f g_\theta, \tag{53}$$

and from equations (45), (48) to (51)

$$(1/r^*)\,\phi_\theta = 2f g_{r^*}. \tag{54}$$

To obtain simple relations for the boundary conditions we introduce new independent variables $\bar{r}$, Y so that

$$\theta = \theta(\bar{r}, Y),$$

$$r^* = \bar{r}$$

and

$$\frac{\partial}{\partial \theta} = \frac{1}{\theta_Y}\frac{\partial}{\partial Y},$$

$$\frac{\partial}{\partial r^*} = \frac{\partial}{\partial \bar{r}} - \frac{\theta_{\bar{r}}}{\theta_Y}\frac{\partial}{\partial Y}.$$

We choose

$$Y = g$$

so that

$$g_\theta = \frac{1}{\theta_Y},$$

$$g_{r^*} = -\frac{\theta_{\bar{r}}}{\theta_Y}.$$

This change of independent variables is permissible if and only if g_θ and θ_Y are neither zero nor infinite. As in §2, we consider only this restricted problem. The differential equations (52) to (54) take the form:

$$\bar{r}\theta_Y = f_{\bar{r}}, \tag{55}$$

$$\bar{r}\theta_Y \phi_{\bar{r}} = -2f[1 + \bar{r}^2(\theta_{\bar{r}})^2], \tag{56}$$

$$\phi_Y = -2f\bar{r}\theta_{\bar{r}}. \tag{57}$$

† There is some freedom in the choice of Ψ_1 and Ψ_2 and therefore in f and g. If

$$\Psi_2 = G(g)$$

then

$$\Psi_1 = h^2 f \frac{1}{dG/dg}$$

satisfies the equations (46) to (48).

We need not follow the paths of the individual particles therefore the value of Ψ_1, i.e. of f, is of no direct interest. However, we do not eliminate the function f but retain it as an auxiliary function for solving the equations.

The static pressure for constant total head can be determined from Bernoulli's equation:

$$\frac{p}{\rho}+\frac{\partial \Phi}{\partial t}+\tfrac{1}{2}(v_r^2+v_\theta^2) = F(t).$$

With the above equations this can be written as

$$-\frac{p}{\rho} = \left[\left(\frac{dh}{dt}\right)^2+h\frac{d^2h}{dt^2}\right][\phi+\tfrac{1}{2}\bar{r}^2]+\frac{1}{2}\left(\frac{dh}{dt}\right)^2\left[-\bar{r}^2+\frac{\phi_{\bar{r}}^2}{1+(\bar{r}\theta_{\bar{r}})^2}\right]-F(t). \quad (58)$$

In the $(\bar{r}, Y)$-plane the flow field is represented by the semi-infinite strip $0 \leqslant Y \leqslant \gamma$, $0 < \bar{r}$, where the region near the centre of the core is represented by the neighbourhood of $\bar{r} = 0$. For two points on opposite faces of the vortex sheet the values of θ differ by 2π; therefore the geometric boundary condition reads

$$\theta(\bar{r},\gamma)-\theta(\bar{r},0) = -2\pi. \quad (59)$$

For the second boundary condition (that the pressure is continuous across the sheet), we obtain from equation (58) the relation

$$\left[1+h\frac{d^2h/dt^2}{(dh/dt)^2}\right]\Delta\phi+\frac{1}{1+(\bar{r}\theta_{\bar{r}})^2}(\phi_{\bar{r}})_m\Delta\phi_{\bar{r}} = 0. \quad (60)$$

3.2. *An asymptotic expansion for the velocity field*

We restrict ourselves to time laws of the form

$$h(t) = kt^{1/m}, \quad (61)$$

where k and m are constants and

$$1+h\frac{d^2h/dt^2}{(dh/dt)^2} = 2-m.$$

An asymptotic solution of equations (55) to (57), (59), (60) has been found by expanding the functions $\theta(\bar{r}, Y)$, $\bar{r}^{2(1-m)}\phi(\bar{r}, Y)$, $\bar{r}^{-2}f(\bar{r}, Y)$ into power series of $R = \pi\bar{r}^m/C$ where the coefficients are functions of Y only and C is an arbitrary constant.

For $0 < m < 2$ and $m \neq 1$, the first terms read

$$\frac{\theta}{\pi} = \frac{1}{mR}-2\frac{Y}{\gamma}+\left[\frac{4-m^2}{2m}\left(2\frac{Y}{\gamma}-1\right)^2+\frac{(2-m)(4-2m-m^2)}{3m^2}\right]R+\ldots, \quad (62)$$

$$\frac{\phi}{C^2} = \bar{r}^{2(1-m)}\left\{\frac{1}{2(m-1)}-\left(2\frac{Y}{\gamma}-1\right)R\right.$$

$$\left.+\left[\frac{2-m}{m}\left(2\frac{Y}{\gamma}-1\right)^2+\frac{(1-m)(2-m)}{3m}-\frac{1}{2\pi^2}\right]R^2+\ldots\right\}. \quad (63)$$

For the special case $m = 1$, the functions $\bar{\theta}(\bar{r}, Y)$, $\phi(\bar{r}, Y) + C^2 \log R$ and $f(\bar{r}, Y)$ were expanded in power series with respect to $R = \bar{r}\pi/C$. The result reads:

$$\frac{\theta}{\pi} = C_1 + (1/R) - A + (\tfrac{3}{2}A^2 + \tfrac{1}{3})R - (\tfrac{4}{3}A^3 - \tfrac{4}{3}A)R^2 - \left(\tfrac{25}{72}A^4 + \tfrac{5}{18}A^2 + \frac{5}{6\pi^2}A^2 - \frac{11}{90} + \frac{1}{9\pi^2}\right)R^3 + \ldots, \tag{64}$$

$$\frac{\phi}{C^2} = C_2 - \log R - AR + \left(A^2 - \frac{1}{2\pi^2}\right)R^2 + \left[-\frac{A^3}{6} + A\right]R^3 - \left[\tfrac{14}{9}A^4 - \tfrac{8}{9}A^2 + \frac{A^2}{3\pi^2} - \frac{11}{90} + \frac{1}{3\pi^2}\right]R^4 + \ldots, \tag{65}$$

with
$$A = 2\frac{Y}{\gamma} - 1,$$

C_1 and C_2 are arbitrary constants.

The vortex sheet is thus given by

$$\theta_s = \frac{C}{\bar{r}} + \pi(C_1 - 1) + \frac{11}{6}\frac{\pi^2}{C}\bar{r} - (\tfrac{181}{360}\pi^4 + \tfrac{17}{18}\pi^2)\frac{1}{C^3}\bar{r}^3 + O(\bar{r}^5) \tag{66}$$

and the potential difference across the sheet by

$$\frac{\Delta\Phi}{t} = -2\pi C\bar{r} + \frac{5}{3}\frac{\pi^3}{C}\bar{r}^3 - (\tfrac{203}{180}\pi^2 + \tfrac{7}{3})(\pi^3/C^3)\bar{r}^5 + O(\bar{r}^7). \tag{67}$$

For the velocity components we obtain† the relations:

$$v_r = \frac{dh}{dt}\left[\bar{r} + \frac{\phi_{\bar{r}}}{1 + (\bar{r}\theta_{\bar{r}})^2}\right] = -A\frac{\pi}{C}\bar{r}^2 - (A^2 + \tfrac{2}{3})\frac{\pi^2}{C^2}\bar{r}^3 + \ldots, \tag{68}$$

$$v_\theta = \frac{dh}{dt}\frac{\bar{r}\theta_{\bar{r}}\phi_{\bar{r}}}{1 + (\bar{r}\theta_{\bar{r}})^2} = +C + A\pi\bar{r} - \left(\frac{A^2}{2} - \frac{1}{3}\right)\frac{\pi^2}{C}\bar{r}^2 - \left(\tfrac{2}{3}A^3 + \frac{A}{\pi^2}\right)\frac{\pi^3}{C^2}\bar{r}^3 + \ldots. \tag{69}$$

It is of course possible to use equation (64) to express A as a function of $\theta - \theta_s(R)$ and R and then to write v_r and v_θ as functions of $\bar{r}$ and θ.

We have stated before that the time-dependent problem for $m = 1$ is equivalent to the conical problem treated by slender theory; we have only to substitute $r/(kx)$ for $\bar{r} = r/(kt)$ (see e.g. Mangler & Smith 1959; Küchemann & Weber 1965).

Maskell has also examined the asymptotic behaviour of the conical leading-edge vortex sheet by slender-body theory and given some results at the I.U.T.A.M. Symposium on Vortex Motions at Ann Arbor in July 1964 (see Smith

† If the centre of the vortex core were not at the origin of the co-ordinate system then we would add the velocity components produced by a uniform flow which is parallel to the line joining the centre of the vortex core and the origin of the co-ordinate system.

1966*b*). He has derived the leading term for the derivatives of $\Delta\Phi$ and of r_s/x with respect to θ where r_s gives the distance of a point on the sheet from the centre of the sheet. The result for $\Delta\Phi$ is the same as given in this paper. For r_s his result reads:

$$\frac{dr_s/x}{d\theta} = -\frac{k_2}{\theta^2}(1+q_1\sin 2\theta - p_1\cos 2\theta)+\ldots,$$

where k_2, q_1 and p_1 are constants. The important difference from the solution of the present report is the appearance of the sinusoidal terms. We have excluded such a type of expansion when we subjected the coefficients in the expansions to certain restrictions.

When introducing a more general type of expansion, we have also tried to avoid the restriction implied in the assumption $\theta_Y \neq 0, \neq \infty$ by choosing as independent variables ξ, η, e.g. the functions $\phi(r,\theta)$ and $g(r,\theta)$. It is not yet known whether it is possible to solve the resulting differential equations for the coefficients in closed form.

A different approach to the asymptotic solution for a slender core was made by Mangler & Sells (1967) who obtained (by conformal mapping) a solution with more degrees of freedom. The leading terms agree with the present cases. The free parameters cannot be determined locally but are used to fit the core to the external flow past the wing and the outer turns of the vortex sheets.

4. Discussion of the results

It is of interest to compare the results derived with and without the slenderness assumption. In order to compare the differential equations, we introduce the variable $X = -\log\bar{r}$ into (55) to (57). Equation (55) reads then

$$\theta_Y = -e^{2X}f_X \tag{70}$$

and equations (56) and (57) become identical with (18) and (19). The important difference between equation (70) and (17) is the factor ϕ on the left-hand side, since ϕ tends to infinity for $X\to\infty$.

The geometric boundary conditions equations (20) and (59) are the same. In order to compare the pressure conditions, we introduce for conical flow the arc length elements ds and $d\sigma$ defined by

$$(ds)^2 = (dR)^2+R^2(d\sigma)^2,$$
$$(d\sigma)^2 = (d\psi)^2+(\sin\psi\, d\theta)^2.$$

Along a streamline Y = const. we have

$$d\sigma = \frac{1}{\cosh X}(1+(\theta_X)^2)^{\frac{1}{2}}dX.$$

We can therefore write the pressure condition of the conical problem, equation (24), in the form

$$\phi_m\Delta\phi+\frac{d\phi_m}{d\sigma}\frac{d\Delta\phi}{d\sigma} = 0. \tag{71}$$

Introducing the arc length $d\sigma$ along the cross section of the sheet in a plane x = const., for which

$$d\sigma = d\bar{r}(1+(\bar{r}\theta_{\bar{r}})^2)^{\frac{1}{2}},$$

we may write the pressure condition for slender theory, equation (60) with $h = t$, in the form

$$\Delta\phi + \frac{d\phi_m}{d\sigma}\frac{d\Delta\phi}{d\sigma} = 0. \tag{72}$$

We note that the difference between equations (71) and (72) is a factor ϕ in the first term.

We note that in slender theory the circumferential velocity, equation (69), tends towards a finite value whilst the circumferential velocity derived without the slenderness assumption, equation (41), has a logarithmic infinity at the centre of the core.

Since the flow considered in slender theory is a two-dimensional flow free of sources, the mean value of the radial velocity taken along a circle $r/x =$ const. is zero. For conical flow, the mean value of the radial velocity taken along a circle $r =$ const. in a plane $x =$ const. need not vanish [according to equation (40) it is $-\frac{1}{2}C^2(r/x) + O((r/x)^3)$], since the mass entering the circle can escape in the x-direction. To obtain an approximate solution of the conical problem, Roy (1966) has suggested introducing a fictitious source distribution in the 'pseudo-transverse' (i.e. pseudo-two-dimensional) flow in a plane $x =$ const. in order to account for this three-dimensional effect.

According to equation (51) the potential function Φ in slender conical theory is related to ϕ by the equation

$$\Phi = x\left[\phi\left(\frac{r}{x}, \theta\right) + \frac{1}{2}\left(\frac{r}{x}\right)^2\right].$$

The streamwise velocity component v_x is therefore (to a first order) equal to ϕ. We note from equation (65) that the velocity component parallel to the axis of the vortex core calculated by slender theory tends to infinity in the same way as when it is calculated without the assumption of slenderness, equation (39).

For the vorticity distribution along the sheet $\gamma = d|\Delta\phi|/d\sigma$ we obtain for the slender solution

$$\gamma = \frac{2\pi}{k}\frac{r}{x} + O\left(\left(\frac{r}{x}\right)^3\right)$$

and for the conical solution

$$\gamma = 2\pi C^2 \frac{r}{x}\left(-\log\frac{r}{x} + \delta + \tfrac{1}{2}\right)^{\frac{1}{2}} + O\left(\left(\frac{r}{x}\right)^3\right).$$

The first term of the slender theory was derived in Mangler & Smith (1959) by a somewhat incomplete argument.

If we compare the derivative

$$\frac{d\theta_s}{d(r/x)} = -\frac{Ck}{(r/x)^2} + \dots,$$

from equation (66), with the value in equation (42), we note that for the conical sheet $|d\theta_s/d(r/x)|$ increases less steeply with decreasing r/x than for the slender sheet, i.e. the spiral is not quite so tightly wound. The slender theory result agrees again with that of Mangler & Smith (1959).

Summarizing, we can say that the introduction of the slenderness assumption has not changed the main character of the asymptotic behaviour near the centre of the vortex sheet. Both theories produce tightly wound spiral sheets for which the vorticity distribution tends to zero at the centre.

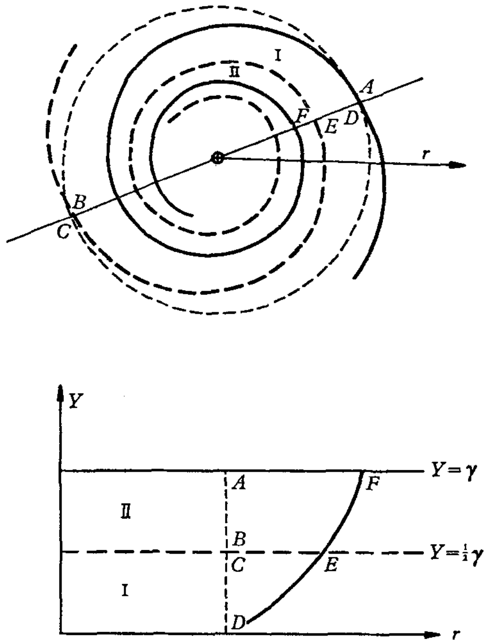

FIGURE 2. Notation for double branched vortex core.

Both theories give logarithmically infinite velocities at the centre. The occurrence of these infinite velocities raises of course some doubts about the relevance of results for incompressible flow. However, for the flow past delta wings at low speed we can expect from the work of Brown (1965) that compressibility effects are important only for a very narrow region of the core. Brown has investigated the compressible axially symmetric conical flow with distributed vorticity and shown that for small values of the Mach number at the edge of the core the flow can be treated as incompressible except for a narrow subcore. In a similar investigation about the effects of viscosity Hall (1961) has shown that the flow can be approximated by an inviscid flow except for a narrow subcore. A comparison with experimental results (see e.g. Küchemann & Weber (1965)) shows that an inviscid incompressible theory can predict some of the important features of the flow.

5. Double branched vortex cores

With the method derived for a time-dependent vortex sheet, we can also determine the flow field past two (or more) similar vortex sheets which have a common centre and a common time law $h(t)$. Stream functions Ψ_1 and Ψ_2 can again be introduced such that both sheets are represented by curves

$$\Psi_2(r/h(t), \theta) = g(r^*, \theta) = \text{const.}$$

We assume again that g_θ does not vanish, so that between the sheets, g is for constant r^* a continuous monotonically varying function. We may choose $Y = g = 0$ to represent the inside face and $Y = \gamma$ the outside face of one sheet. For the position of the second sheet any value of Y in the interval $0 < Y < \gamma$ can be chosen. We select the particular case of a symmetrical two-branched sheet (see figure 2) by choosing $Y = \frac{1}{2}\gamma$ and a constant value for the leading term in θ_Y. We denote by the suffix I the functions for $0 < Y < \frac{1}{2}\gamma$ and by the suffix II the functions for $\frac{1}{2}\gamma < Y < \gamma$. The differential equations are the same as before and with the time law given by equation (61) the boundary conditions read:

$$\theta_{\mathrm{II}}(\bar{r},\gamma) - \theta_{\mathrm{I}}(\bar{r},0) = -2\pi,$$

$$\theta_{\mathrm{II}}(\bar{r},\tfrac{1}{2}\gamma) - \theta_{\mathrm{I}}(\bar{r},\tfrac{1}{2}\gamma) = 0,$$

$$2(2-m)\,[\phi_{\mathrm{II}}(\bar{r},\gamma) - \phi_{\mathrm{I}}(\bar{r},0)]\,[1+(\bar{r}\theta_{\bar{r}})^2] + [\phi_{\mathrm{II}\bar{r}}(\bar{r},\gamma) + \phi_{\mathrm{I}\bar{r}}(\bar{r},0)]\,[\phi_{\mathrm{II}\bar{r}}(\bar{r},\gamma) - \phi_{\mathrm{I}\bar{r}}(\bar{r},0)] = 0,$$

$$2(2-m)\,[\phi_{\mathrm{II}}(\bar{r},\tfrac{1}{2}\gamma) - \phi_{\mathrm{I}}(\bar{r},\tfrac{1}{2}\gamma)]\,[1+(\bar{r}\theta_{\bar{r}})^2] + [\phi_{\mathrm{II}\bar{r}}(\bar{r},\tfrac{1}{2}\gamma) + \phi_{\mathrm{I}\bar{r}}(\bar{r},\tfrac{1}{2}\gamma)]\,[\phi_{\mathrm{II}\bar{r}}(\bar{r},\tfrac{1}{2}\gamma) - \phi_{\mathrm{I}\bar{r}}(\bar{r},0)] = 0.$$

For $0 < m < 2$ and $m \neq 1$, the first terms of an asymptotic solution, corresponding to equations (62) and (63) read:

$$\frac{\theta}{\pi} = \frac{1}{mR} - 2\frac{Y}{\gamma} + \left[\frac{4-m^2}{2m}\left(2\frac{Y}{\gamma}-1\right)^2 \pm \frac{4-m^2}{m}K\left(2\frac{Y}{\gamma}-1\right) + \frac{(2-m)(4-2m-m^2)}{3m^2} - K\frac{(2-m)(6+7m-7m^2-2m^3)}{3m^3} + K^2\frac{(2-m)^2(6+2m-m^2)}{2m^2} - K^3\frac{(2-m)^3(2+m)}{2m^3}\right]R + \dots, \tag{73}$$

$$\frac{\phi}{C^2} = \bar{r}^{2(1-m)}\left\{\frac{1}{2(m-1)} - \left(2\frac{Y}{\gamma} - 1 \pm K\right)R + \left[\frac{2-m}{m}\left(2\frac{Y}{\gamma}-1\right)^2 \pm \frac{2-m}{m}2K\left(2\frac{Y}{\gamma}-1\right) + \frac{(1-m)(2-m)}{3m} - \frac{1}{2\pi^2} - K\frac{(2-m)(6-2m-4m^2+m^3)}{3m^2} + K^2\frac{(2-m)^2(4+m-m^2)}{2m} - K^3\frac{(2-m)^3(2+m)}{2m^2}\right]R^2 + \dots\right\}, \tag{74}$$

where the upper sign applies in the interval $0 < Y < \frac{1}{2}\gamma$ and the lower sign in the interval $\frac{1}{2}\gamma < Y < \gamma$. The constant K determines the potential difference across the second sheet. If we consider only the terms of lowest order, then we note that the ratio between the potential differences for the second sheet and the first sheet is $K/(1-K)$. Equations (73) and (74) are applicable in the case $m = 1$, except that the term $\bar{r}^{2(1-m)}/2(m-1)$ in equation (74) is replaced by $-\log R$.

6. Conclusions

Some asymptotic solutions for incompressible flow near the centre of a rolled-up vortex sheet have been obtained for conical steady flow and for time-dependent two-dimensional flow. Once the rather special form of the expansions is decided, the solutions are unique and the number of free constants is very small. In both cases the shape of the sheet is that of a tightly wound spiral. For conical flow the axial and circumferential velocity components tend to infinity in the same way as for axially symmetrical rotational conical flow (Hall 1961).

The results have been used to show the differences between the solutions derived with and without the assumptions of slender theory. The circumferential velocity tends, with decreasing distance from the axis, logarithmically to infinity for the conical solution and to a finite value for the slender solution. The velocity component normal to a circle around the centre of the spiral has for the conical solution a finite mean value which tends to zero with the distance from the centre; for the slender solution this mean value is always zero. The axial velocity component tends to infinity for both solutions.

Comparisons of an asymptotic core solution based on inviscid flow with experiments made in a real flow are very difficult. Küchemann & Weber (1965) gave some examples which showed that the tightly rolled spiral and the large axial velocity components predicted by the calculation seem to be consistent with experiments.

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